Numerical Method for Solution of Particulate Flow Equations

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Theme

THE equations describing the motion of a medium consisting of viscous fluid and suspended solid particles are nonlinear, partial differential equations, and are always coupled even in the case of incompressible fluid phase. Analytical solutions could be obtained for frozen and equilibrium flow regimes of a few problems with simplified models for the forces of bulk interaction between the two phases. If the nonequilibrium particulate flow is to be studied, numerical computational methods have to be used. A numerical method to analyze the unsteady two-dimensional motion of incompressible viscous gas and suspended solid particles is presented. This method is used to study the particulate flow due to the impulsive motion of an infinite flat plate in an otherwise stationary suspension.

Contents

The motion of the fluid and solid particles is governed by the equations of continuity and momentum of the gas, the equations of linear and angular momentum of particles, and the equation of conservation of mass of particles. The equation of angular momentum of the suspension was derived by the authors[1] and was used to determine the stress tensor in the particulate flow. The equations of motion are solved to determine the following particulate flow properties; the fluid velocity, the particle concentration, and the translational and rotational speeds of the solid particles throughout the flowfield. The solid particle translational and rotational velocities are generally different from those of the gas phase, depending on the flow regime. In the frozen or near frozen regime, the difference between the translational velocity of the two phases which is referred to as the slip velocity is very large. The particulate flow is said to be in the near equilibrium and equilibrium regimes when the slip velocity is small and approaching zero. The non-equilibrium flow regime corresponds to the intermediate range of slip velocity.

The equations of motion are solved in nondimensional form, the normalized suspension flow variables are defined as

$$t^* = t/\tau_t, \quad u^* = u/U, \quad u_p{}^* = u_p/U, \quad \omega^* = [\tau_t/(Re)^{1/2}]\omega$$
$$y^* = [(Re)^{1/2} y/U\tau_t], \quad v^* = (Re)^{1/2} v/u; \quad v_p{}^* = (Re)^{1/2} v_p/U \tag{1}$$

where $Re = U^2\tau_t/\nu$. The time t is normalized with respect to the characteristic time of particle translational motion τ_t where $\tau_t = (d^2/18\nu)(\rho_p/\rho)$, u and v are the gas velocity components in the x and y directions, u_p and v_p are the particle velocity components in the x and y directions, ω is the solid particle rotational velocity, ρ the gas density, ρ_p the solid particle material density, d the solid particle diameter, and U is a characteristic velocity of the flow.

Before solving the governing equations, a transformation of the independent variables is used. Self-similar solutions cannot be obtained in the case of particular flows due to the characteristic relaxation times associated with the translational and rotational motions of the particles. A transformation that leads to a self-similar solution in nonparticulate flows is used to eliminate the discontinuity that would otherwise exist in the initial gas velocity profile in the physical plane. The transformation also makes it unnecessary to add more mesh points in the direction normal to the flow as the thickness of the momentum boundary layer increases. For the example considered here, the following transformation is used:

$$\eta = y^*/2(t^*)^{1/2} \tag{2}$$

The forces and torque of bulk interaction between the two phases depend generally on the range of some parameters of the particulate flow under consideration. The drag force and torque due to the difference between the translational and rotational velocities of the particles and the fluid, as well as the lift force due to the translational motion of the particles in the shear flow are considered. The expressions used in the present example correspond to dilute suspensions and to low slip Reynolds number, ignoring the gravity forces. The slip Reynolds number is based on the slip velocity between the particles and the gas and the solid particle diameter. The numerical method presented here can be used however for any other range of flow parameters if the corresponding expression of the forces and torque of bulk interaction are used.

The equations that govern the unsteady two-dimensional motion of the suspension in the transformed plane (η, t^*) are

$$\frac{\partial[(1-\chi)v^*]}{\partial\eta} + \frac{\partial[\chi v_p{}^*]}{\partial\eta} = 0 \tag{3}$$

$$\frac{\partial\chi}{\partial t^*} - \frac{\eta}{2t^*}\frac{\partial\chi}{\partial\eta} + \frac{1}{2(t^*)^{1/2}}\left(v_p{}^*\frac{\partial\chi}{\partial\eta} + \chi\frac{\partial v_p{}^*}{\partial\eta}\right) = 0 \tag{4}$$

$$(1-\chi)\left[\frac{\partial u^*}{\partial t^*} + \left(\frac{v^*}{2(t^*)^{1/2}} - \frac{\eta}{2t^*}\right)\frac{\partial u^*}{\partial\eta}\right] =$$
$$\frac{1}{4t^*}[1+1.5\chi]\frac{\partial^2 u^*}{\partial\eta^2} + \frac{1.5}{4t^*}\frac{\partial\chi}{\partial\eta}\frac{\partial u^*}{\partial\eta} + \frac{3}{2(t^*)^{1/2}}\frac{\partial(\chi\omega^*)}{\partial\eta} -$$
$$\frac{\rho_p}{\rho}\chi G(u^* - u_p{}^*) + \frac{3.23}{2\pi}\chi\left(-\frac{1}{(Re\, t^*)^{1/2}}\frac{\rho_p}{\rho}\frac{\partial u^*}{\partial\eta}\right)^{1/2} \times$$
$$(v^* - v_p{}^*) \tag{5}$$

$$\frac{\partial u_p{}^*}{\partial t^*} + \left(\frac{v_p{}^*}{2(t^*)^{1/2}} - \frac{\eta}{2t^*}\right)\frac{\partial u_p{}^*}{\partial\eta} = G(u^* - u_p{}^*) -$$
$$\frac{3.23}{\pi}\left(\frac{\rho}{\rho_p}\right)^{1/2} Re^{-1/4}\left(-\frac{1}{4(t^*)^{1/2}}\frac{\partial u^*}{\partial\eta}\right)^{1/2}(v^* - v_p{}^*) \tag{6}$$

$$\frac{\partial v_p{}^*}{\partial t^*} + \left(\frac{v_p{}^*}{2(t^*)^{1/2}} - \frac{\eta}{2t^*}\right)\frac{\partial v_p{}^*}{\partial\eta} = G(v^* - v_p{}^*) +$$
$$\frac{3.23}{\pi}\left(\frac{\rho}{\rho_p}\right)^{1/2} Re^{3/4}\left(-\frac{1}{4(t^*)^{1/2}}\frac{\partial u^*}{\partial\eta}\right)^{1/2}(u^* - u_p{}^*) \tag{7}$$

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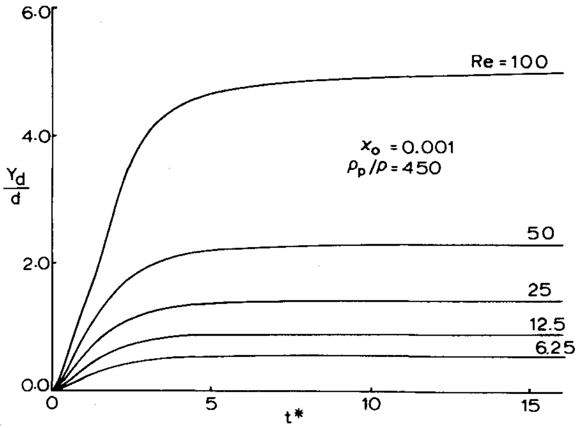

Fig. 1 Effect of the Reynolds number on the demixed region.

$$\frac{\partial \omega^*}{\partial t^*} + \left(\frac{v_p^*}{2(t^*)^{1/2}} - \frac{\eta}{2t^*}\right)\frac{\partial \omega^*}{\partial \eta} = -\frac{\tau_t}{\tau_r}\left[\omega^* + \frac{1}{4(t^*)^{1/2}}\frac{\partial u^*}{\partial \eta}\right] \quad (8)$$

where

$$G = \left\{1 + \frac{9}{16}\left(\frac{2\rho}{\rho_p}\right)^{1/2}[Re(u^* - u_p^*)^2 + (v^* - v_p^*)^2]^{1/2}\right\} \times \{1 + 1.5(2\chi)^{1/2} + 3.75\chi\}$$

χ is the particle concentration by volume and $\tau_r = (d^2/60\nu)(\rho_p/\rho)$ is the characteristic time of particle rotational motion.

At the wall, the gas phase should satisfy the no slip condition. If the flat plate impulsive velocity is U, then: at $\eta = 0$, $u^* = 1$, $v^* = 0$. There is no boundary condition equivalent to the no slip condition for the solid particle phase. The particles are free to slide on or collide with the plate. Both phases should satisfy some freestream conditions which are the undisturbed flow conditions in the present case. At $\eta = \infty$; $u^* = 0$, $u_p^* = 0$, $v^* = 0$, $v_p^* = 0$, $\chi = \chi_o$.

The initial conditions for the particle phase are the frozen conditions. In the present example of an impulsive flat plate the initial conditions are $u_p^* = 0$, $v_p^* = 0$; $\omega^* = 0$. The initial conditions for the gas phase are not so straightforward. The main routine used for calculating u^* at any later time is used to iterate for the profile at $t^* = 0$ from a crudely guessed initial profile, with $u^* = 1$ at $\eta = 0$ and zero elsewhere.

Implicit second-order finite difference schemes are used to solve the equations of motion of the gas-particle suspension. The equation of conservation of the gas phase in the x-direction, Eq. (5), is a second-order parabolic equation that is solved for u^*. Equations (4) and (6–8) are first-order parabolic equations that involve derivatives of χ, u_p^*, v_p^*, and ω^*. These equations represent the conservation of mass of the particles, the conservation of linear momentum of the particles in the x and y-direction, and the conservation of angular momentum of the particles. The equation of conservation of mass of the suspension, Eq. (3), is integrated numerically to determine v^*.

The criterion used for the test of convergence of all particle properties was that the difference between the last calculation and the previous iteration of the maximum value of the property profile be below a specified tolerance. For the gas velocity u^*, the convergence was determined based on the comparison of the values of the velocity gradient at the plate, $(\partial u^*/\partial \eta)|_{\eta=0}$, calculated from two successive iterations.

The main routine allows for the addition of mesh points in the η direction, if the difference between the values of any flow property at the two outmost positions exceeds a prescribed tolerance. The time step size was increased with the progress of motion, as the rates of change of flow properties decreased with particles approaching equilibrium conditions.

Most of the numerically determined values of particulate flow properties, obtained using the present analysis, were reported by the authors.[1,2] One of the flow variables that was determined numerically is the demixed region Y_d that developed next to the plate. In this region, no particles exist due to their migration away from the wall. The distance Y_d was calculated as explained,[2] using the continuum approach, and by evaluating the distance traveled in the y-direction by the particles that were initially at the wall. The results obtained using both approaches were in agreement. Figure 1 shows that the demixed particle region increases with time from zero initially approaching asymptotically the final equilibrium value that depends on the Reynolds number. Although this phenomenon has been reported in the experimental data of particulate flow in pipes and channels, it has not been predicted in any of several theoretical studies of equilibrium or near equilibrium regimes. Only by studying the nonequilibrium regime numerically can the demixed particle region be determined, since in both frozen and equilibrium flow regimes the particle translation velocity component normal to the wall is zero.

In conclusion, the numerical method presented is useful for studying particulate flow problems involving frozen, equilibrium, and nonequilibrium flow regimes, as well as problems involving all or more than one of these regimes. Although the solution was for incompressible flow, the same approach could be used to solve compressible particulate flows if the energy exchange between the phases is known.

References

[1] Hamed, A. and Tabakoff, W., "Some Effects Caused by Solid Particles in Flows," *AIAA Journal*, Vol. 12, May 1974, pp. 581, 582.

[2] Hamed, A. and Tabakoff, W., "Solid Particle Demixing in a Suspension Flow of Viscous Gas," *Fluid Mechanics of Mixing*, edited by E. M. Uram and V. W. Goldschmidt, ASME, New York, 1973, pp. 101–115.